JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 3, May–June 1992

Robust Time-Optimal Control of Uncertain Flexible Spacecraft

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A new approach for computing time-optimal open-loop control inputs for uncertain flexible spacecraft is developed. In particular, the single-axis, rest-to-rest maneuvering problem of flexible spacecraft in the presence of uncertainty in model parameters is investigated. Robust time-optimal control inputs are obtained by solving a parameter optimization problem subject to robustness constraints. A simple dynamical system with a rigid-body mode and one flexible mode is used to illustrate the concept.

I. Introduction

THIS paper is concerned with the problem of computing open-loop control inputs for flexible spacecraft, robotic manipulators, and pointing systems in space, which are often required to maneuver as quickly as possible without significant structural vibrations during and/or after a maneuver.

A standard, time-optimal control approach to such a problem requires an accurate mathematical model and, thus, the resulting solution is often sensitive to variations in model parameters. For this reason, an open-loop time-optimal controller is seldom used in practice. Consequently, the development of a "robustified" open-loop approach for a rapid maneuver without significant structural vibrations is of current research interest.[1-3] Other open-loop approaches[4-6] attempt to find a smooth continuous forcing function (e.g., a versine function) that begins and ends with zero slope. The basic idea behind such approaches is that a smooth control input without sharp transitions is less likely to excite structural modes during maneuvers.

In this paper, a new approach is developed for computing time-optimal control inputs for the single-axis, rest-to-rest maneuvering problem of flexible spacecraft in the presence of structural mode frequency uncertainty. A parameter optimization problem, where the objective function to be minimized is the maneuvering time, is formulated with additional constraints for robustness with respect to structural parameter uncertainty. The resulting robust time-optimal solution is a multiswitch bang-bang control that can be implemented for spacecraft equipped with on-off reaction jets.[7] This result further confirms that most open-loop approaches, which utilize a smooth continuous control input so that structural modes are less likely to be excited, do not fully utilize the available control energy in performing a robust time-optimal maneuver.

The paper is organized as follows. In Sec. II, the rest-to-rest maneuvering constraints for multiswitch bang-bang inputs are derived with some discussion on the previous results of Refs. 8–10 on the number of switchings for the time-optimal solution. The standard, time-optimal control problem is then transformed into a constrained parameter optimization problem. In Sec. III, robustness constraints are derived and incorporated with the parameter optimization problem formulated in Sec. II. A simple dynamical system with a rigid-body mode and one flexible mode, shown in Fig. 1, is used to illustrate the concept, and the robust time-optimal solution for a case with a single two-sided control (case 1) is discussed. In Sec. IV, the same example with two one-sided control inputs (case 2) is further investigated.

Received March 11, 1991; presented as Paper 91-2646 at the AIAA Guidance, Navigation, and Control Conference, New Orleans, LA, Aug. 12–14, 1991; revision received Aug. 25, 1991; accepted for publication Sept. 25, 1991.

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II. Time-Optimal Rest-to-Rest Maneuver

Problem Formulation

Consider a linear model of flexible spacecraft described by

$$M\ddot{x} + Kx = Gu \tag{1}$$

where x is a generalized displacement vector, M a mass matrix, K a stiffness matrix, G a control input distribution matrix, and u a control input vector.

In this section, we consider a case with a scalar control input $u(t)$ bounded as

$$-1 \leq u \leq 1 \tag{2}$$

Equation (1) can be transformed into the decoupled modal equations:

$$\begin{aligned} \ddot{y}_1 + \omega_1^2 y_1 &= \phi_1 u \\ \ddot{y}_2 + \omega_2^2 y_2 &= \phi_2 u \\ &\vdots \\ \ddot{y}_n + \omega_n^2 y_n &= \phi_n u \end{aligned} \tag{3}$$

where y_i is the ith modal coordinate, ω_i the ith modal frequency, ϕ_i the ith modal gain, and n the number of modes considered in control design.

The problem is to find the control input that minimizes the performance index

$$J = \int_0^{t_f} \mathrm{d}t = t_f$$

subject to Eqs. (2) and (3) and given boundary conditions.

The time-optimal control problem of a linear controllable system has a unique solution, which is bang-bang control with a finite number of switches.[10] For a spring-mass dynamical system with n degrees of freedom, the time-optimal bang-bang solution for a rest-to-rest maneuver has, in most cases, $(2n-1)$ switches,[8,9] and the solution is symmetric about $t_f/2$. That is, for a case with $(2n-1)$ switches, we have

$$t_j = t_{2n} - t_{2n-j}; \qquad j = 1, \cdots, n \tag{4}$$

where $t_{2n} = t_f$.

A bang-bang input with $(2n-1)$ switches can then be represented as

$$u(t) = \sum_{j=0}^{2n} B_j u_s(t - t_j) \tag{5}$$

where B_j is the magnitude of a unit step function $u_s(t)$ at t_j. This function can be characterized by its switch pattern as

$$B = \{B_0, \quad B_1, \quad B_2, \quad \cdots, \quad B_{2n}\}$$
$$T = \{t_0, \quad t_1, \quad t_2, \quad \cdots, \quad t_{2n}\}$$

where B represents a set of B_j with $B_0 = B_{2n} = \pm 1$ and $B_j = \pm 2$ for $j = 1, \cdots, 2n-1$; T represents a set of switching times $(t_1, \cdots, t_{2n-1})$ and the initial and final times ($t_0 = 0$ and $t_f = t_{2n}$).

Rest-to-Rest Maneuver Constraints

Consider the rigid-body mode equation with $\omega_1 = 0$

$$\ddot{y}_1 = \phi_1 u \tag{6}$$

with the rest-to-rest maneuvering boundary conditions

$$\begin{aligned} y_1(0) &= 0, \qquad y_1(t_f) \neq 0 \\ \dot{y}_1(0) &= 0, \qquad \dot{y}_1(t_f) = 0 \end{aligned} \tag{7}$$

Substituting Eq. (5) into Eq. (6) and solving for the time response of the rigid-body mode, we get

$$y_1(t \geq t_f) = \frac{\phi_1}{2} \sum_{j=0}^{2n} (t_f - t_j)^2 B_j \tag{8}$$

The rest-to-rest maneuvering constraint for the rigid-body mode can then be written as

$$\frac{\phi_1}{2} \sum_{j=0}^{2n} (t_f - t_j)^2 B_j - y_1(t_f) = 0 \tag{9}$$

Consider now the structural modes described by

$$\ddot{y}_i + \omega_i^2 y_i = \phi_i u; \qquad i = 2, \cdots, n \tag{10}$$

with the corresponding boundary conditions for the rest-to-rest maneuver:

$$\begin{aligned} y_i(0) &= 0, \qquad y_i(t_f) = 0 \\ \dot{y}_i(0) &= 0, \qquad \dot{y}_i(t_f) = 0 \end{aligned} \tag{11}$$

for each flexible mode.

Substituting Eq. (5) into the ith structural mode equation and solving for the time response for $t \geq t_f$, we get

$$\begin{aligned} y_i(t) &= -\frac{\phi_i}{\omega_i^2} \sum_{j=0}^{2n} B_j \cos \omega_i (t - t_j) \\ &= -\frac{\phi_i}{\omega_i^2} \Bigg[\cos \omega_i (t - t_n) \sum_{j=0}^{2n} B_j \cos \omega_i (t_j - t_n) \\ &\quad + \sin \omega_i (t - t_n) \sum_{j=0}^{2n} B_j \sin \omega_i (t_j - t_n) \Bigg] \end{aligned} \tag{12}$$

It can be shown that the following constraint equation for each mode

$$\sum_{j=0}^{2n} B_j \sin \omega_i (t_j - t_n) = 0 \tag{13}$$

is always satisfied for any bang-bang input that is symmetric about the midmaneuver time t_n. Consequently, we have the following flexible mode constraints for no residual structural vibration [i.e., $y_i(t) = 0$ for $t \geq t_f$]:

$$\sum_{j=0}^{2n} B_j \cos \omega_i (t_j - t_n) = 0 \tag{14}$$

for each flexible mode.

Parameter Optimization Problem

For a spring-mass system of n degress of freedom, the time-optimal solution represented by Eq. (5) has the $(2n-1)$ unknown switching times and the final time t_f to be determined. The time-optimal control problem can now be formulated as a constrained parameter optimization problem as follows.

Determine a control of the form given by Eq. (5) that minimizes the final time t_f subject to

$$\frac{\phi_1}{2} \sum_{j=0}^{2n} (t_f - t_j)^2 B_j - y_1(t_f) = 0 \tag{15a}$$

$$\begin{aligned} \sum_{j=0}^{2n} B_j \cos \omega_i (t_j - t_n) &= 0; \qquad i = 2, \cdots, n \\ t_j &> 0; \qquad j = 1, \cdots, 2n \end{aligned} \tag{15b}$$

where $t_f = t_{2n}$.

Remark: Note that ϕ_i $(i = 2, \cdots, n)$ do not appear in Eqs. (15); that is, the optimal solution to this problem is independent of the flexible mode shapes. In other words, the time-optimal control input is independent of actuator location for a system described by Eqs. (3) with a scalar input.

Standard optimization packages can be used to obtain the solution of the previous optimization problem. The major advantage of the proposed approach, compared to other direct numerical optimization approaches employed in Refs. 11–14 for the time-optimal control problem, is that some robustness constraints with respect to plant parameter uncertainty can be augmented easily. This subject will be discussed in detail in Sec. III.

Sufficient Conditions for Optimality

Equations (15) are necessary conditions for time-optimal control, and sufficient conditions for optimality can be checked as follows.

Let the costate vector corresponding to the modal state vector $[y_1, \dot{y}_1, \omega y_2, \dot{y}_2, \cdots, \omega_n y_n, \dot{y}_n]^T$ be defined as

$$\lambda(t) = [p_1(t), \quad q_1(t), \quad \cdots, \quad p_n(t), \quad q_n(t)]^T \tag{16}$$

It is shown in Ref. 8 that at midmaneuver we have

$$\lambda(t_n) = [p_1(t_n), \quad 0, \quad p_2(t_n), \quad 0, \quad \cdots, \quad p_n(t_n), \quad 0]^T \tag{17}$$

and, thus, the costate vector can be solved as

$$\begin{aligned} p_1(t) &= p_1(t_n) \\ q_1(t) &= -(t - t_n) p_1(t_n) \\ p_i(t) &= p_i(t_n) \cos \omega_i (t - t_n) \\ q_i(t) &= -p_i(t_n) \sin \omega_i (t - t_n) \end{aligned} \tag{18}$$

for each flexible mode. Then $p_i(t_n)$, for $i = 1, \cdots, n$, can be found from the following n linear equations:

$$1 + \phi_1 p_1(t_n) t_n + \sum_{i=2}^{n} \phi_i p_i(t_n) \sin \omega_i t_n = 0 \tag{19}$$

$$\phi_1 p_1(t_n)(t_j - t_n) + \sum_{i=2}^{n} \phi_i p_i(t_n) \sin \omega_i (t_j - t_n) = 0 \tag{20}$$

where $j = n+1, \cdots, 2n-1$.

The solution obtained by minimizing t_f subject to Eqs. (15) becomes time optimal provided that

$$S(t) = -\phi_1 p_1(t_n)(t - t_n) - \sum_{i=2}^{n} \phi_i p_i(t_n) \sin \omega_i (t - t_n) \neq 0 \tag{21}$$

for $t \in (t_n, t_{2n})$ and $t \neq t_j$; $j = n+1, \cdots, 2n$, and $S(t)$ represents the switching function.

Remark: If modal frequencies are rational multiples of each other and the fundamental frequency ω_2 satisfies the following relationship

$$\omega_2 = 2l\pi\sqrt{\frac{\phi_1}{y_1(t_f)}}, \qquad l = 1,2,\cdots \tag{22}$$

then the time-optimal solution has only one switch and is equivalent to the solution of a "rigidized" case.

Example: Case 1 with a Scalar Control Input

Consider a simple example, shown in Fig. 1, which is a generic representation of a flexible spacecraft with a rigid-body mode and one flexible mode. Case 1 with a scalar control input $u(t)$ is considered here. The equations of motion are

$$m_1\ddot{x}_1 + k(x_1 - x_2) = u_1 = u \tag{23a}$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) = u_2 = 0 \tag{23b}$$

where x_1 and x_2 are the positions of body 1 and body 2, respectively, and the nominal parameters are $m_1 = m_2 = k = 1$ with appropriate units, and time is in seconds.

The boundary conditions for a rest-to-rest maneuver are given as

$$\begin{aligned} x_1(0) = x_2(0) = 0, &\qquad x_1(t_f) = x_2(t_f) = 1 \\ \dot{x}_1(0) = \dot{x}_2(0) = 0, &\qquad \dot{x}_1(t_f) = \dot{x}_2(t_f) = 0 \end{aligned} \tag{24}$$

The modal equations are

$$\ddot{y}_1 = u/2 \tag{25a}$$

$$\ddot{y}_2 + \omega^2 y_2 = u/2 \tag{25b}$$

where $\omega = \sqrt{2}$ rad/s is the nominal flexible mode frequency. The corresponding boundary conditions for modal coordinates are

$$\begin{aligned} y_1(0) = y_2(0) = 0, &\qquad y_1(t_f) = 1, \qquad y_2(t_f) = 0 \\ \dot{y}_1(0) = \dot{y}_2(0) = 0, &\qquad \dot{y}_1(t_f) = \dot{y}_2(t_f) = 0 \end{aligned} \tag{26}$$

Since there are three switches, the time-optimal switch pattern for the given boundary conditions is represented as

$$B = \{B_0, \; B_1, \; B_2, \; B_3, \; B_4\}$$

$$= \{1, \; -2, \; 2, \; -2, \; 1\}$$

$$T = \{t_0, \; t_1, \; t_2, \; t_3, \; t_4\}$$

with the symmetry conditions

$$t_4 = 2t_2$$

$$t_3 = 2t_2 - t_1$$

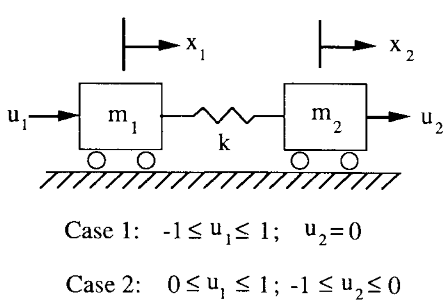

Fig. 1 Generic model with a rigid-body mode and one flexible mode.

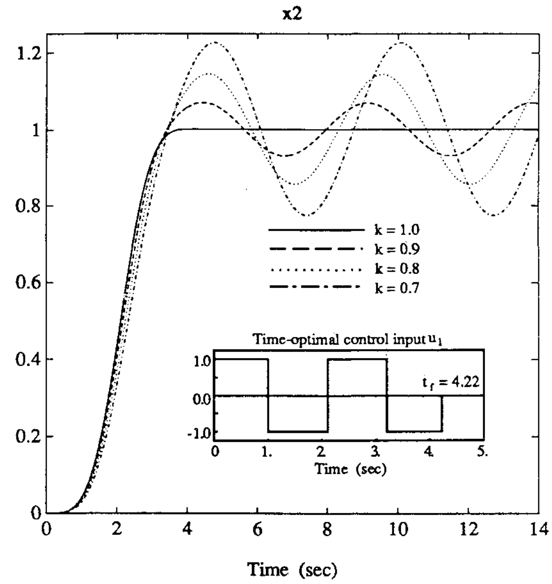

Fig. 2 Responses to time-optimal control input (case 1).

The time-optimal control problem is then formulated as the following constrained minimization problem:

$$\min J = 2t_2 \tag{27}$$

subject to

$$2 + 2t_1^2 + t_2^2 - 4t_1t_2 = 0$$

$$1 - 2\cos\omega(t_2 - t_1) + \cos\omega t_2 = 0$$

$$t_1, \; t_2 > 0$$

A standard IMSL FORTRAN subroutine was used to obtain a solution as $t_1 = 1.003$ and $t_2 = 2.109$. The computed solution satisfies the optimality conditions of Eq. (21); i.e., the switching function vanishes only at $t = t_1$, t_2, and t_3. Thus, the solution obtained via Eq. (27) is indeed time optimal and it can be expressed as

$$u(t) = u_s(t) - 2u_s(t - 1.003) + 2u_s(t - 2.109)$$

$$- 2u_s(t - 3.215) + u_s(t - 4.218) \tag{28}$$

The time responses of x_2 to the time-optimal control input are shown in Fig. 2 for four different values of k. It can be seen that the resulting responses are sensitive to variations in the model parameter k.

III. Robust Time-Optimal Control

As discussed in the preceding section, a standard, time-optimal control approach requires an accurate mathematical model, and thus, the resulting solution is often sensitive to plant modeling uncertainty.

In this section, a new approach, expanding on the approach introduced in Sec. II, is developed for computing time-optimal control inputs for the single-axis, rest-to-rest maneuvering problem of flexible spacecraft in the presence of structural mode frequency uncertainty. A parameter optimization problem, where the objective function to be minimized is the maneuvering time, is formulated with additional constraints for robustness with respect to the structural frequency uncertainty. The resulting robustified, time-optimal solution is a multiswitch bang-bang control and is thus implementable for spacecraft equipped with on-off reaction jets.[7]

Robustness Constraints

By taking the derivative of Eqs. (12) with respect to ω_i, we get

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}\omega_i} = \frac{\phi_i}{\omega_i^2}\cos\omega_i\left(t - \frac{t_f}{2}\right)\sum_{j=0}^{2n}\left(t_j - \frac{t_f}{2}\right)B_j \sin\omega_i\left(t_j - \frac{t_f}{2}\right) \tag{29}$$

for each flexible mode. Letting $\mathrm{d}y_i(t)/\mathrm{d}\omega_i = 0$ for all $t \geq t_f$, we have

$$\sum_{j=0}^{2n}\left(t_j - \frac{t_f}{2}\right)B_j \sin\omega_i\left(t_j - \frac{t_f}{2}\right) = 0; \qquad i = 2,\cdots,n \tag{30}$$

which is called the first-order robustness constraints.

Similarly, taking the derivative of Eqs. (12) r_i times with respect to ω_i results in r_ith-order robustness constraints for each flexible mode as follows:

$$\sum_{j=0}^{2n}\left(t_j - \frac{t_f}{2}\right)^m B_j \sin\omega_i\left(t_j - \frac{t_f}{2}\right) = 0 \quad \text{for } m = 1,3,\cdots \leq r_i \tag{31a}$$

$$\sum_{j=0}^{2n}\left(t_j - \frac{t_f}{2}\right)^m B_j \cos\omega_i\left(t_j - \frac{t_f}{2}\right) = 0 \quad \text{for } m = 2,4,\cdots \leq r_i \tag{31b}$$

There are total r robustness constraints for $(n-1)$ flexible modes, where

$$r = \sum_{i=2}^{n} r_i \tag{32}$$

If these robustness constraints are included in the constrained minimization problem formulation described by Eqs. (15), the number of switches in the bang-bang control input, in most cases, must be increased to match the number of the constraint equations. Because of the symmetric nature of the rest-to-rest maneuvering problem, adding one robustness constraint will require, at least, two more switches.

Robust Time-Optimal Control

If r robustness constraints are considered for a flexible spacecraft of n modes, the corresponding robustified bang-bang control input becomes

$$u(t) = \sum_{j=0}^{2(n+r)} B_j u_s(t - t_j) \tag{33}$$

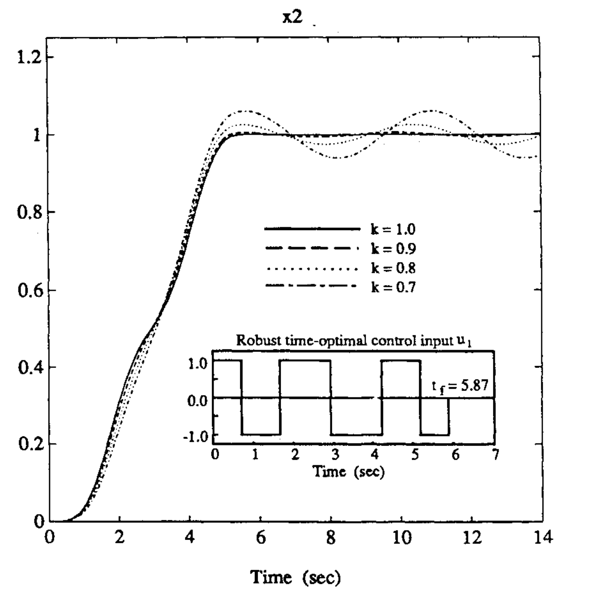

Fig. 3 Responses to robust time-optimal control input (case 1).

which has $2(n+r)$ unknown switching times. Because of the symmetry property of the optimal solution for the rest-to-rest maneuvering problem, we have

$$t_j = t_{2(n+r)} - t_{2(n+r)-j}; \qquad j = 1,\cdots,n+r \tag{34}$$

Therefore, there are only $(n+r)$ unknowns to be determined in Eq. (33). These unknowns can be determined by minimizing t_f subject to the $(n+r)$ constraint equations: one positioning constraint for the rigid-body mode, $(n-1)$ no-vibration constraints, and r robustness constraints.

Although many theoretical issues (e.g., the uniqueness of the optimal solution) need to be explored, a solution can be obtained by solving the following constrained parameter optimization problem:

$$\min J = t_f = t_{2(n+r)} \tag{35}$$

subject to

$$\frac{\phi_1}{2}\sum_{j=0}^{2(n+r)}\left[t_{2(n+r)} - t_j\right]^2 B_j - y_1(t_f) = 0$$

$$\sum_{j=0}^{2(n+r)} B_j \cos\omega_i(t_j - t_{n+r}) = 0$$

$$\sum_{j=0}^{2(n+r)} (t_j - t_{n+r})^m B_j \sin\omega_i(t_j - t_{n+r}) = 0 \qquad \text{for } m = 1,3,\cdots \leq r_i$$

$$\sum_{j=0}^{2(n+r)} (t_j - t_{n+r})^m B_j \cos\omega_i(t_j - t_{n+r}) = 0 \qquad \text{for } m = 2,4,\cdots \leq r_i$$

$$t_j > 0; \qquad j = 1,\cdots,2(n+r)$$

for each flexible mode. The resulting bang-bang control input, which has $2r$ more switches than the time-optimal bang-bang solution of Sec. II, is called a robust (or robustified) time-optimal solution in this paper.

Example: Case 1 with a Scalar Control Input

For case 1, the time-optimal control is a three-switch bang-bang function, but the resulting responses were shown to be sensitive to variations in model parameter k. A robust time-optimal solution of the same problem is now developed as follows. The switching pattern for a case with the first-order robustness constraint is assumed as

$$B = \{B_0, \ B_1, \ B_2, \ B_3, \ B_4, \ B_5, \ B_6\} = \{1, \ -2, \ 2, \ -2, \ 2, \ -2, \ 1\} \tag{36a}$$

$$T = \{t_0, \ t_1, \ t_2, \ t_3, \ t_4, \ t_5, \ t_6\} \tag{36b}$$

with the symmetry conditions

$$t_4 = 2t_3 - t_2$$
$$t_5 = 2t_3 - t_1 \tag{37}$$
$$t_6 = 2t_3$$

The constrained optimization problem with the first-order robustness constraint can be formulated as

$$\min J = t_6 \tag{38}$$

subject to

$$2 + 2t_1^2 - 2t_2^2 - t_3^2 - 4t_1t_3 + 4t_2t_3 = 0 \tag{39a}$$

$$\cos \omega t_3 - 2\cos \omega(t_3 - t_1) + 2\cos \omega(t_3 - t_2) - 1 = 0 \quad (39b)$$

$$t_3 \sin \omega t_3 - 2(t_3 - t_1)\sin \omega(t_3 - t_1) + 2(t_3 - t_2)\sin \omega(t_3 - t_2) = 0 \quad (39c)$$

$$t_1,\ t_2,\ t_3 > 0$$

A robust time-optimal solution with five switches can be found as

$$t_1 = 0.7124, \quad t_2 = 1.6563$$
$$t_3 = 2.9330, \quad t_4 = 4.2097 \quad (40)$$
$$t_5 = 5.1536, \quad t_6 = 5.8660$$

The time responses of x_2 to this robustified time-optimal control input are shown in Fig. 3 for four different values of k. It can be seen that the resulting responses are less sensitive to parameter variations, compared to the responses to the ideal, time-optimal control input as shown in Fig. 2. Performance robustness has been increased at the expense of the increased maneuvering time of 5.866 s, comparing to the ideal minimum time of 4.218 s. It is, however, emphasized that simply prolonging the maneuver time does not help to reduce residual structural vibrations caused by modeling uncertainty.

Remarks: An impulse-sequence shaping technique, developed by Singer and Seering,[1,2] was employed by Wie and Liu[3] to preshape the ideal, time-optimal control input given in Eq. (28). For example, the two-impulse preshaped bang-bang command was obtained as

$$u(t) = 0.5u_s(t) - u_s(t-1.003) + u_s(t-2.109) + 0.5u_s(t-2.221) - u_s(t-3.215) - u_s(t-3.224) + 0.5u_s(t-4.218) + u_s(t-4.330) - u_s(t-5.436) + 0.5u_s(t-6.439) \quad (41)$$

This preshaped input takes values of ± 1.0 and ± 0.5 and the resulting response becomes less sensitive to flexible mode frequency variations as demonstrated in Ref. 3.

For case 1, the time response to the time-optimal input of Eq. (28), the robust time-optimal input of Fig. 3, and the preshaped inputs in Ref. 3 can be compared as shown in Figs. 4 and 5. Comparisons of the maneuvering times of four different control schemes are illustrated in Fig. 4. Parameter robustness with respect to spring constant variations is compared in Fig. 5. From these figures, it is evident that the proposed approach of this paper provides a faster and more robust maneuver than other robustified feed-forward approaches. Also, unlike other approaches in Refs. 1–6, the resulting solution of our new approach is a multiswitch bang-bang control that can be implemented for spacecraft with on-off reaction jets.

In the next section, we will consider a case with two one-sided control inputs in order to further explore a time-optimal actuator placement problem.

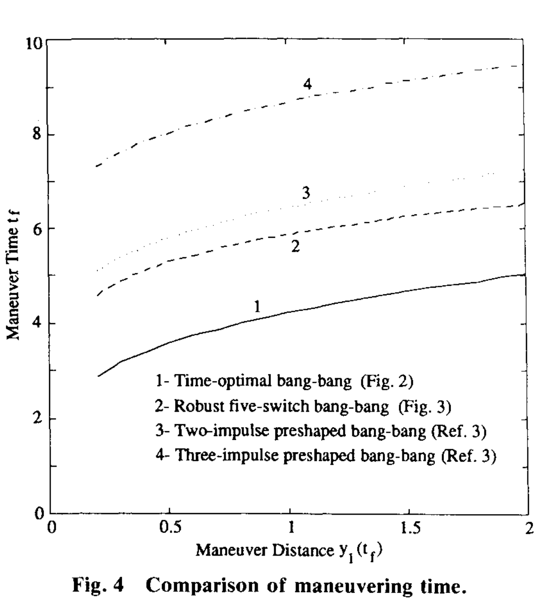

Fig. 4 Comparison of maneuvering time.

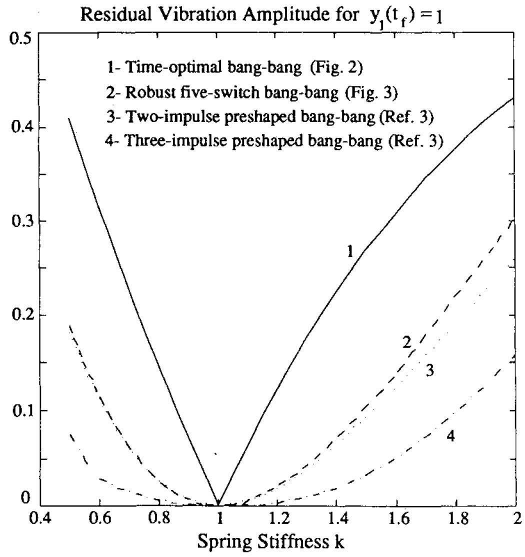

Fig. 5 Comparison of parameter sensitivity.

IV. Case with Two One-Sided Control Inputs

Problem Formulation

Consider case 2, illustrated in Fig. 1, with two one-sided control inputs bounded as

$$0 \le u_1 \le +1 \quad (42a)$$

$$-1 \le u_2 \le 0 \quad (42b)$$

Since the control inputs are one sided, each control input for the time-optimal solution need not be an odd function about the midmaneuver time. Thus, the problem with one-sided control inputs becomes more difficult to solve than the standard problem with two-sided control inputs, and many theoretical issues (e.g., the uniqueness and structure of time-optimal solutions) need further investigation.

For case 2, the modal equations of the system with nominal parameter values are

$$\ddot{y}_1 = \tfrac{1}{2}(u_1 + u_2) \quad (43a)$$

$$\ddot{y}_2 + \omega^2 y_2 = \tfrac{1}{2}(u_1 - u_2) \quad (43b)$$

where $\omega = \sqrt{2}$ rad/s is the nominal flexible mode frequency.

For the control input constraint given by Eqs. (42), the control inputs can be expressed as

$$u_1 = \sum_{j=0,2,4,\cdots}^{N-1} \left[u_s(t-t_j) - u_s(t-t_j-\Delta_j)\right] \quad (44a)$$

$$u_2 = -\sum_{j=1,3,5,\cdots}^{N} \left[u_s(t-t_j) - u_s(t-t_j-\Delta_j)\right] \quad (44b)$$

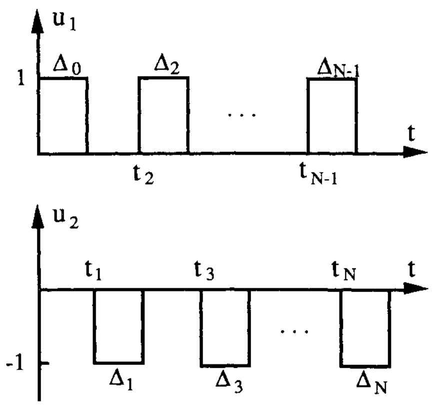

Fig. 6 Pulse sequences (case 2).

which is in the form of one-sided pulse sequences, as shown in Fig. 6. The jth pulse starts at t_j and ends at $(t_j+\Delta_j)$. Because of the symmetric nature of the rest-to-rest maneuvering problem, we assume that u_1 and u_2 have the same number of pulses, $(N+1)/2$, where N is defined as in Fig. 6.

Substituting Eqs. (44) into Eq. (43a) and solving for the time response of the rigid-body mode, we get

$$y_1(t \geq t_f) = \tfrac{1}{4} \sum_{j=0}^{N} (-1)^j \left[2t\Delta_j - 2t_j\Delta_j - \Delta_j^2\right] \tag{45}$$

For the desired boundary condition, $y_1(t \geq t_f) = 1$, the following constraint must hold:

$$\sum_{j=0}^{N} (-1)^j \Delta_j = 0 \tag{46}$$

The positioning constraint for the rigid-body mode then becomes

$$\sum_{j=0}^{N} (-1)^j \left[2t_j\Delta_j + \Delta_j^2\right] + 4 = 0 \tag{47}$$

Substituting Eqs. (44) into Eq. (43b) and solving for the time response of the flexible mode, we get

$$y_2(t) = -\tfrac{1}{4} \cos \omega t \sum_{j=0}^{N} \left[\cos \omega t_j - \cos \omega(t_j+\Delta_j)\right]$$

$$-\tfrac{1}{4} \sin \omega t \sum_{j=0}^{N} \left[\sin \omega t_j - \sin \omega(t_j+\Delta_j)\right] \tag{48}$$

for $t \geq t_f$.

Also, rest-to-rest maneuvering requires $y_2(t)=0$ for $t \geq t_f$; i.e., we have

$$\sum_{j=0}^{N} \left[\cos \omega t_j - \cos \omega(t_j+\Delta_j)\right] = 0 \tag{49a}$$

$$\sum_{j=0}^{N} \left[\sin \omega t_j - \sin \omega(t_j+\Delta_j)\right] = 0 \tag{49b}$$

which become the no-vibration constraints for the rest-to-rest maneuvering problem.

Time-Optimal Control

Let the time-optimal control inputs for the rest-to-rest maneuver problem be of the form

$$u_1 = u_s(t) - u_s(t-\Delta)$$

$$u_2 = -u_s(t-t_1) + u_s(t-t_1-\Delta)$$

where each input has a single pulse with the same pulse width of Δ, t_1 is defined as shown in Fig. 6, and the maneuver time $t_f = t_1 + \Delta$.

The rest-to-rest maneuver constraints can be obtained from Eqs. (47) and (49) as

$$t_f - (2/\Delta) - \Delta = 0 \tag{50a}$$

$$\sin(\omega t_f/2) + \sin\left[\omega(\Delta - t_f/2)\right] = 0 \tag{50b}$$

which can be combined as

$$\sin(\omega\Delta/2)\cos\left[\omega/(2\Delta)\right] = 0 \tag{51}$$

The time-optimal solution can then be obtained by solving the constrained minimization problem:

$$\min J = t_f = (2/\Delta) + \Delta \tag{52}$$

subject to the constraint given by Eq. (51).

The solution of this problem can be found as

$$\Delta = 0.9003 \qquad t_f = 3.1218$$

The time responses of x_2 to the time-optimal control inputs are shown in Fig. 7 for four different values of k. The maneuver time and control-on time are, respectively, 3.12 and 1.8 s. As expected, the resulting responses are sensitive to parameter variations.

Remark: A most interesting feature of this solution is that the overall input shape shown in Fig. 7 is of a bang-off-bang type, resulting in the control-on time of 1.8 s, which is different from the maneuver time t_f of 3.12 s. For case 1 and a rigidized case,[3] both the maneuver time and control-on time are 4.218 and 2.828 s, respectively. Therefore, the actuator configuration for case 2 is considered to be optimal in the sense of minimizing both the maneuver time and control-on time.

Robust Time-Optimal Control

Similar to case 1 in Sec. II, we now consider the robustification of the time-optimal solution obtained in the preceding section.

Letting the derivative of Eq. (48) with respect to ω be zero, we get

$$\frac{dy_2}{d\omega} = -\tfrac{1}{4} \sin \omega t \sum_{j=0}^{N} \left[t_j \cos \omega t_j - (t_j+\Delta_j)\cos \omega(t_j+\Delta_j)\right]$$

$$+\tfrac{1}{4} \cos \omega t \sum_{j=0}^{N} \left[t_j \sin \omega t_j - (t_j+\Delta_j)\sin \omega(t_j+\Delta_j)\right]$$

$$= 0 \tag{53}$$

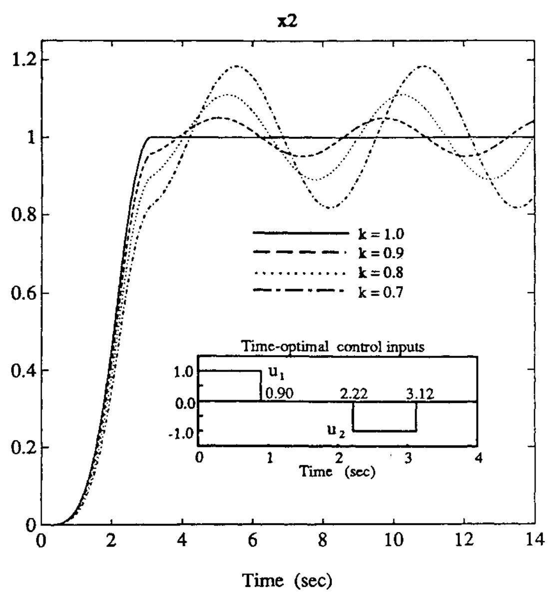

Fig. 7 Responses to time-optimal control inputs (case 2).

For this derivative to be zero for arbitrary $t \geq t_f$, we must have

$$\sum_{j=0}^{N} \left[t_j \cos \omega t_j - (t_j + \Delta_j) \cos \omega(t_j + \Delta_j) \right] = 0 \tag{54a}$$

$$\sum_{j=0}^{N} \left[t_j \sin \omega t_j - (t_j + \Delta_j) \sin \omega(t_j + \Delta_j) \right] = 0 \tag{54b}$$

which are called the first-order robustness constraints.

Taking the derivative of Eq. (48) r times with respect to ω, we get the rth-order robustness constraint equations for input pulse sequences as follows:

$$\sum_{j=0}^{N} \left[(t_j)^m \cos \omega t_j - (t_j + \Delta_j)^m \cos \omega(t_j + \Delta_j) \right] = 0 \tag{55}$$

$$\sum_{j=0}^{N} \left[(t_j)^m \sin \omega t_j - (t_j + \Delta_j)^m \sin \omega(t_j + \Delta_j) \right] = 0 \tag{56}$$

for $m = 1, 2, \cdots, r$.

As an example, we consider the first-order robustness constraint, incorporated with the rest-to-rest maneuver constraints, to construct robust time-optimal pulse sequences. Assuming that each input has two pulses, we can represent the control inputs as follows:

$$u_1 = u_s(t) - u_s(t - \Delta_0) + u_s(t - t_2) - u_s(t - t_2 - \Delta_2) \tag{57a}$$

$$u_2 = -u_s(t - t_1) + u_s(t - t_1 - \Delta_1) - u_s(t - t_3) + u_s(t - t_3 - \Delta_3) \tag{57b}$$

in which we have seven unknowns to be determined, and t_j and Δ_j are defined as shown in Fig. 6.

The robust time-optimal solution can then be obtained by solving the constrained parameter optimization problem

$$\min J = t_3 + \Delta_3 \tag{58}$$

subject to

$$\Delta_0 - \Delta_1 + \Delta_2 - \Delta_3 + \Delta_4 = 0$$

$$\sum_{j=0}^{3} (-1)^j \left[2 t_j \Delta_j + \Delta_j^2 \right] + 4 = 0$$

$$\sum_{j=0}^{3} \left[\cos \omega t_j - \cos \omega(t_j + \Delta_j) \right] = 0$$

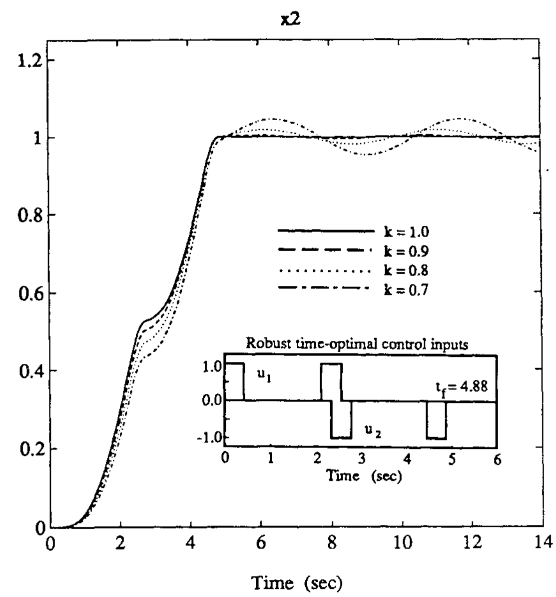

Fig. 8 Responses to robust time-optimal control inputs (case 2).

$$\sum_{j=0}^{3} \left[\sin t_j - \sin \omega(t_j + \Delta_j) \right] = 0$$

$$\sum_{j=0}^{3} \left[t_j \cos \omega t_j - (t_j + \Delta_j) \cos \omega(t_j + \Delta_j) \right] = 0$$

$$\sum_{j=0}^{3} \left[t_j \sin \omega t_j - (t_j + \Delta_j) \sin \omega(t_j + \Delta_j) \right] = 0$$

$$\Delta_j \geq 0; \qquad j = 0, 1, 2, 3$$

$$t_1, \; t_2, \; t_3 > 0$$

The solution to this problem can be obtained as

$$\begin{aligned} t_0 &= 0.0000, & \Delta_0 &= 0.4274 \\ t_1 &= 2.3357, & \Delta_1 &= 0.4329 \\ t_2 &= 2.1132, & \Delta_2 &= 0.4329 \\ t_3 &= 4.4544, & \Delta_3 &= 0.4274 \end{aligned} \tag{59}$$

The time responses of x_2 to the robust time-optimal control inputs are shown in Fig. 8 for four different values of k. It is seen that the robustness has been increased at the expense of the increased maneuvering time of 4.882 s, compared to the ideal minimum time of 3.122 s. However, note that the control-on time is only 1.721 s, compared to the control-on time of 1.8 s of the ideal, time-optimal solution.

V. Conclusions

A new approach to the robust time-optimal control of uncertain flexible spacecraft has been investigated. The unique feature of the proposed approach is the fairly straightforward incorporation of the robustness constraints into a standard parameter optimization problem, where the objective function to be minimized is the maneuvering time. The case with two one-sided control inputs has shown an interesting feature from the viewpoint of optimal actuator placement for minimizing both the maneuver time and control-on time.

Acknowledgments

This research was supported by the NASA Johnson Space Center (JSC) through the University of Houston at Clear Lake. The authors would like to express special thanks to Kenneth Cox and John Sunkel at NASA JSC for sponsoring this research. The authors would also like to thank Karl Bilimoria for his thorough review of this paper and for his technical assistance.

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